New conditional symmetries and exact solutions of reaction–diffusion systems with power diffusivities

2008 J. Phys. A: Math. Theor. 41 185208

(http://iopscience.iop.org/1751-8121/41/18/185208)

J. Phys. A: Math. Theor. **41** (2008) 185208 (14pp) doi:10.1088/1751-8113/41/18/185208

New conditional symmetries and exact solutions of reaction–diffusion systems with power diffusivities

Roman Cherniha[1,2] **and Oleksii Pliukhin**[1]

[1] Institute of Mathematics, Ukrainian National Academy of Sciences, Tereshchenkivs'ka Street 3, Kyiv 01601, Ukraine
[2] Lesya Ukrayinka Volyn' National University, 13, Volia Avenue, Lutsk 43025, Ukraine

E-mail: cherniha@imath.kiev.ua and pliukhin@imath.kiev.ua

Received 21 November 2007, in final form 14 March 2008
Published 21 April 2008
Online at stacks.iop.org/JPhysA/41/185208

Abstract
A wide range of new Q-conditional symmetries for reaction–diffusion systems with power diffusivities are constructed. The relevant non-Lie ansätze to reduce the reaction–diffusion systems to ODE systems and examples of exact solutions are obtained. The relation of the solutions obtained with the development of spatially inhomogeneous structures is discussed.

PACS numbers: 02.20.–a, 02.30.Jr, 05.45.–a

1. Introduction

In 1952, A C Turing published the remarkable paper [1], in which a revolutionary idea about the mechanism of morphogenesis (the development of structures in an organism during its life) has been proposed. Roughly speaking, his idea says that the diffusion process can interact with the chemical reaction process in such a way that this can stimulate the development and growth of different forms and structures in an organism. Moreover this differentiation is often impossible without diffusion, i.e., only one may destabilize the spatially homogeneous structures. This effect is known as the Turing instability and mathematically leads to systems of reaction–diffusion equations with nonlinearities of the special form [2, 3]. Since 1952 the reaction–diffusion systems have been extensively studied by means of different mathematical methods, including group-theoretical methods. The majority of the attention was paid to the investigation of the two-component RD systems of the form

$$\begin{aligned} U_t &= [D^1(U)U_x]_x + F(U, V), \\ V_t &= [D^2(V)V_x]_x + G(U, V), \end{aligned} \tag{1}$$

where $U = U(t, x)$ and $V = V(t, x)$ are two unknown functions representing the densities of cells and chemicals, respectively, $F(U, V)$ and $G(U, V)$ are two given functions describing the interaction between them and the environment, the functions $D^1(U)$ and $D^2(V)$ are the relevant diffusivities (usually they are assumed to be constants or power functions) and the subscripts t and x denote differentiation with respect to these variables. At present, one can claim that all possible Lie symmetries of (1) with the constant diffusivities are completely described in [4–6], while in [7–9] it has been done for the non-constant diffusivities.

The problem of construction of conditional symmetries for (1) is still not solved even in the case of Q-conditional symmetries (non-classical symmetries). Moreover, to the best of our knowledge, there are no such papers devoted to the search for conditional symmetries of the RD system (1). Here, we present some recent results in this direction for the first time. It should be noted that there are many papers devoted to the construction of such symmetries for the scalar nonlinear reaction–diffusion (RD) equations of the form [10–16]

$$U_t = [D(U)U_x]_x + F(U), \tag{2}$$

and (2) with the convective term $B(U)U_x$ (here $B(U)$, $D(U)$ and $F(U)$ are arbitrary smooth functions) [17–19].

It is well known that conditional symmetries can be applied for finding exact solutions of the relevant equations, which are not obtainable by the classical Lie method. Moreover, the solutions obtained in such a way may have a physical or a biological interpretation (see, e.g., the examples in [18–21]). In this paper, we will demonstrate this using Q-conditional symmetries of nonlinear RD systems of the form (1).

The paper is organized as follows. In the second section, we present new Q-conditional symmetries of the RD system

$$\begin{aligned} U_t &= (U^k U_x)_x + F(U, V), \\ V_t &= (V^l V_x)_x + G(U, V), \end{aligned} \tag{3}$$

which is the most important subcase of (1) from the applicability point of view [2, 22]. The determining equations for constructing Q-conditional symmetries of the system (3) are derived and the main theorem presenting these symmetries in explicit form is proved.

In the third section, the Q-conditional symmetries obtained are applied for reducing the corresponding RD systems to the systems of ordinary differential equations (ODE). A nonlinear RD system with the specified functions $F(U, V)$ and $G(U, V)$ is considered in detail. We show that this system with the correctly specified coefficients may lead to the Turing instability. Finally, we present some conclusions.

2. Main result

It is well known that the main difficulty arising in the search for Q-conditional symmetries is to solve the so-called system of determining equations, which is nonlinear and overdetermined. Roughly speaking, one arrives at the contradiction trying to construct Q-conditional symmetries of a given nonlinear PDE (system of PDEs) because there is the need to solve the new nonlinear PDE system, which is usually much more cumbersome. This problem arises even in the case of linear PDE and this was the reason why Bluman and Cole in their pioneering work [23] were unable to describe all Q-conditional symmetries in an explicit form even for the linear-heat equation. In the recent paper [19], a short historical review is devoted to this problem and different kinds of non-Lie symmetry.

Here, we construct the system of determining equations to find Q-conditional symmetry operators of the form

$$Q = \partial_t + \xi(t, x, U, V)\partial_x + \eta^1(t, x, U, V)\partial_U + \eta^2(t, x, U, V)\partial_V, \tag{4}$$

where ξ, η^1 and η^2 are unknown functions for the RD system (3) and solve one under the relevant restrictions. Note that we search for purely conditional symmetry operators, which cannot be reduced to Lie symmetry operators. We also remind the reader that each Q-conditional symmetry operator is determined up to equivalent representations generated by multiplying (4) on the arbitrary smooth function $M(t, x, U, V)$. The problem of finding Q-conditional symmetry operators of the form

$$Q = \xi(t, x, U, V)\partial_x + \eta^1(t, x, U, V)\partial_U + \eta^2(t, x, U, V)\partial_V \tag{5}$$

is another challenge and will be treated elsewhere.

First, we remind the reader of the notion of the conditional symmetry following the book [10, section 5.7] (note one was formulated for the first time in [24]).

Definition. *A PDE of the form*

$$S(x_1, \ldots, x_r, U, \underset{1}{U}, \ldots \underset{p}{U}) = 0 \tag{6}$$

(here $U = U(x_1, \ldots, x_r)$ and U_k is the totality of kth-order derivatives) is conditionally invariant under the operator

$$Q = \xi^1(x_1, \ldots, x_r, U)\partial_{x_1} + \cdots + \xi^r(x_1, \ldots, x_r, U)\partial_{x_r} + \eta(x_1, \ldots, x_r, U)\partial_U, \tag{7}$$

where η and ξ^a with $a = 1, \ldots, r$, are smooth functions, if it is invariant (in Lie's sense) under this operator only together with an additional condition of the form

$$S_Q(x_1, \ldots, x_r, U, \underset{1}{U}, \ldots, \underset{q}{U}) = 0, \tag{8}$$

that is, the overdetermined system of equations (6) and (8) is invariant under a Lie group generated by the operator Q. If the additional condition (8) coincides with the equation $Q(U) = 0$, i.e.,

$$\xi^1(x_1, \ldots, x_r, U)U_{x_1} + \cdots + \xi^1(x_1, \ldots, x_r, U)U_{x_r} = \eta(x_1, \ldots, x_r, U)$$

then PDE (6) is Q-conditionally invariant under the operator (7).

Obviously, this definition admits a direct generalization on systems of two PDEs. Of course, if the given system is not in involution then some difficulties can occur, however, system (3) has the involution form. Let us show how the determining equations to find the Q-conditional symmetry operator (4) are obtained.

It turns out that one can apply the local substitution

$$\begin{aligned} u &= U^{k+1}, \qquad k \neq -1, \\ v &= V^{l+1}, \qquad l \neq -1 \end{aligned} \tag{9}$$

to simplify the further computations. Of course, the case $(k+1)(l+1) = 0$ is special and needs a separate investigation. Substitution (9) reduces system (3) and operator (4) to the form

$$\begin{aligned} u_{xx} &= u^m u_t + C^1(u, v), \\ v_{xx} &= v^n v_t + C^2(u, v), \end{aligned} \tag{10}$$

and

$$Q = \partial_t + \xi(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v, \tag{11}$$

where $m = -\frac{k}{k+1} \neq -1, n = -\frac{p}{p+1} \neq -1, C^1(u, v) = -(k+1)F\left(u^{\frac{1}{k+1}}, v^{\frac{1}{l+1}}\right), C^2(u, v) = -(p+1)G\left(u^{\frac{1}{k+1}}, v^{\frac{1}{l+1}}\right), \partial_u = \frac{1}{k+1}U^{-k}\partial_U$ and $\partial_v = \frac{1}{l+1}V^{-l}\partial_V$.

Let us now apply the second prolongation of the operator Q

$$\underset{11}{Q} = Q + \rho_t^1\frac{\partial}{\partial u_t} + \rho_t^2\frac{\partial}{\partial v_t} + \rho_x^1\frac{\partial}{\partial u_x} + \rho_x^2\frac{\partial}{\partial v_x} + \sigma_{tx}^1\frac{\partial}{\partial u_{tx}} + \sigma_{tx}^2\frac{\partial}{\partial v_{tx}} + \sigma_{tt}^1\frac{\partial}{\partial u_{tt}} + \sigma_{tt}^2\frac{\partial}{\partial v_{tt}} + \sigma_{xx}^1\frac{\partial}{\partial u_{xx}} + \sigma_{xx}^2\frac{\partial}{\partial v_{xx}},$$

where coefficients ρ^k and σ^k with the relevant indices are calculated by the known formulae (see, e.g., [10, 25]), to each equation of (10). Since system (10) should be considered as a manifold in the space of independent variables

$$t, x, u, v, u_t, v_t, u_x, v_x, u_{xt}, v_{xt}, u_{tt}, v_{tt}, u_{xx}, v_{xx},$$

we arrive at the invariance condition

$$\begin{aligned} &m\eta^1 u^{m-1}u_t + \eta^1 C_u^1 + \eta^2 C_v^1 + \rho_t^1 u^m = \sigma_{xx}^1,\\ &n\eta^2 v^{n-1}v_t + \eta^2 C_v^2 + \eta^1 C_u^2 + \rho_t^2 v^n = \sigma_{xx}^2. \end{aligned} \tag{12}$$

To obtain the determining equation for the Q-conditional symmetry operator (11), one needs to take into account not only system (10) (it will lead only to the determining equation for Lie symmetry operators) but also two additional conditions

$$u_t + \xi u_x = \eta^1, \qquad v_t + \xi v_x = \eta^2 \tag{13}$$

generated by operator (11). Thus, inserting into (12) the explicit expressions for ρ^k and σ^k, namely,

$$\begin{aligned} \rho_t^1 &= \eta_t^1 + \eta_u^1 u_t + \eta_v^1 v_t - u_x(\xi_t + \xi_u u_t + \xi_v v_t),\\ \rho_t^2 &= \eta_t^2 + \eta_u^2 u_t + \eta_v^2 v_t - v_x(\xi_t + \xi_u u_t + \xi_v v_t),\\ \sigma_{xx}^1 &= \eta_{xx}^1 + 2\eta_{xu}^1 u_x + 2\eta_{xv}^1 v_x + \eta_{uu}^1(u_x)^2 + \eta_{vv}^1(v_x)^2 + 2\eta_{uv}^1 u_x v_x + \eta_u^1 u_{xx} + \eta_v^1 v_{xx}\\ &\quad - u_x(\xi_{xx} + 2\xi_{xu}u_x + 2\xi_{xv}v_x + \xi_{uu}(u_x)^2 + \xi_{vv}(v_x)^2 + 2\xi_{uv}u_x v_x + \xi_u u_{xx} + \xi_v v_{xx})\\ &\quad - 2u_{xx}(\xi_x + \xi_u u_x + \xi_v v_x),\\ \sigma_{xx}^2 &= \eta_{xx}^2 + 2\eta_{xu}^2 u_x + 2\eta_{xv}^2 v_x + \eta_{uu}^2(u_x)^2 + \eta_{vv}^2(v_x)^2 + 2\eta_{uv}^2 u_x v_x + \eta_u^2 u_{xx} + \eta_v^2 v_{xx}\\ &\quad - u_x(\xi_{xx} + 2\xi_{xu}u_x + 2\xi_{xv}v_x + \xi_{uu}(u_x)^2 + \xi_{vv}(v_x)^2 + 2\xi_{uv}u_x v_x + \xi_u u_{xx} + \xi_v v_{xx})\\ &\quad - 2v_{xx}(\xi_x + \xi_u u_x + \xi_v v_x) \end{aligned} \tag{14}$$

and excluding four derivatives u_t, v_t, u_{xx}, v_{xx} using (10) and (13), one arrives at the cumbersome expressions

$$\begin{aligned} &u^m\big(\eta_t^1 + \eta_u^1(\eta^1 - \xi u_x) + \eta_v^1(\eta^2 - \xi v_x) - u_x(\xi_t + \xi_u(\eta^1 - \xi u_x) + \xi_v(\eta^2 - \xi v_x))\big)\\ &\quad + m\eta^1 u^{m-1}(\eta^1 - \xi u_x) + \eta^1 C_u^1 + \eta^2 C_v^1\\ &\quad = \eta_{xx}^1 + 2\eta_{xu}^1 u_x + 2\eta_{xv}^1 v_x + \eta_{uu}^1(u_x)^2 + \eta_{vv}^1(v_x)^2 + 2\eta_{uv}^1 u_x v_x\\ &\quad - u_x(\xi_{xx} + 2\xi_{xu}u_x + 2\xi_{xv}v_x + \xi_{uu}(u_x)^2\\ &\quad + \xi_{vv}(v_x)^2 + 2\xi_{uv}u_x v_x) + ((\eta^1 - \xi u_x)u^m + C^1)\big(\eta_u^1 - 2\xi_x - 3\xi_u u_x - 2\xi_v v_x\big)\\ &\quad + ((\eta^2 - \xi v_x)v^n + C^2)\big(\eta_v^1 - \xi_v u_x\big),\\ &v^n\big(\eta_t^2 + \eta_u^2(\eta^1 - \xi u_x) + \eta_v^2(\eta^2 - \xi v_x) - v_x(\xi_t + \xi_u(\eta^1 - \xi u_x) + \xi_v(\eta^2 - \xi v_x))\big)\\ &\quad + n\eta^2 v^{n-1}(\eta^2 - \xi v_x) + \eta^1 C_u^2 + \eta^2 C_v^2\\ &\quad = \eta_{xx}^2 + 2\eta_{xu}^2 u_x + 2\eta_{xv}^2 v_x + \eta_{uu}^2(u_x)^2 + \eta_{vv}^2(v_x)^2 + 2\eta_{uv}^2 u_x v_x\\ &\quad - v_x(\xi_{xx} + 2\xi_{xu}u_x + 2\xi_{xv}v_x + \xi_{uu}(u_x)^2\\ &\quad + \xi_{vv}(v_x)^2 + 2\xi_{uv}u_x v_x) + ((\eta^2 - \xi v_x)v^n + C^2)\big(\eta_u^2 - 2\xi_x - 3\xi_v v_x - 2\xi_u u_x\big)\\ &\quad + ((\eta^1 - \xi u_x)u^m + C^1)\big(\eta_u^2 - \xi_u v_x\big). \end{aligned} \tag{15}$$

Finally, we take into account that the unknown functions η^1, η^2 and ξ do not depend on the derivatives u_x and v_x and therefore we split two expressions arising in (15) into $(u_x)^3, u_x(v_x)^2, v_x(u_x)^2, u_xv_x, (u_x)^2, (v_x)^2, v_x, u_x$ and $(v_x)^3, v_x(u_x)^2, u_x(v_x)^2, u_xv_x, (u_x)^2, (v_x)^2, v_x, u_x$, respectively. Thus, we arrive at the nonlinear system of determining equations

$$
\begin{aligned}
&(1) \quad \xi_{uu} = \xi_{vv} = \xi_{uv} = 0,\\
&(2) \quad \eta^1_{vv} = 0,\\
&(3) \quad \eta^2_{uu} = 0,\\
&(4) \quad 2\xi\xi_u u^m + \eta^1_{uu} - 2\xi_{xu} = 0,\\
&(5) \quad 2\xi\xi_v v^n + \eta^2_{vv} - 2\xi_{xv} = 0,\\
&(6) \quad \xi\xi_v(u^m + v^n) + 2\eta^1_{uv} - 2\xi_{xv} = 0,\\
&(7) \quad \xi\xi_u(u^m + v^n) + 2\eta^2_{uv} - 2\xi_{xu} = 0,\\
&(8) \quad \xi\eta^1_v(u^m - v^n) + 2\eta^1_{xv} - 2\xi_v C^1 - 2\xi_v\eta^1 u^m = 0,\\
&(9) \quad \xi\eta^2_u(v^n - u^m) + 2\eta^2_{xu} - 2\xi_u C^2 - 2\xi_u\eta^2 v^n = 0,\\
&(10) \quad -m\xi\eta^1 u^{m-1} + (2\xi_u\eta^1 - \xi_t - \xi_v\eta^2 - 2\xi\xi_x)u^m\\
&\qquad + \xi_v\eta^2 v^n + 3\xi_u C^1 + \xi_v C^2 - 2\eta^1_{xu} + \xi_{xx} = 0,\\
&(11) \quad -n\xi\eta^2 v^{n-1} + (2\xi_v\eta^2 - \xi_t - \xi_u\eta^1 - 2\xi\xi_x)v^n\\
&\qquad + \xi_u\eta^1 u^m + 3\xi_v C^2 + \xi_u C^1 - 2\eta^2_{xv} + \xi_{xx} = 0,\\
&(12) \quad m(\eta^1)^2 u^{m-1} + \left(\eta^1_t + \eta^2\eta^1_v + 2\xi_x\eta^1\right)u^m - \eta^2\eta^1_v v^n\\
&\qquad + \eta^1 C^1_u + \eta^2 C^1_v - \eta^1_u C^1 + 2\xi_x C^1 - \eta^1_v C^2 - \eta^1_{xx} = 0,\\
&(13) \quad n(\eta^2)^2 v^{n-1} + \left(\eta^2_t + \eta^1\eta^2_u + 2\xi_x\eta^2\right)v^n - \eta^1\eta^2_u u^m\\
&\qquad + \eta^1 C^2_u + \eta^2 C^2_v - \eta^2_u C^1 + 2\xi_x C^2 - \eta^2_v C^2 - \eta^2_{xx} = 0.
\end{aligned} \tag{16}
$$

to find the coefficients of the operator (11) and the functions C^1, C^2. Equations (1) from this system are easily integrated and lead to

$$\xi = a(t,x)u + b(t,x)v + c(t,x), \tag{17}$$

where a, b and c are arbitrary (at the moment) smooth functions. Substituting (17) into equations (6) and (7) of (16) and taking into account the second and third equations of (16), one arrives at the requirement $a = b = 0$. Thus, equations (2)–(7) of system (16) can be straightforwardly integrated and their general solution takes the form

$$
\begin{aligned}
\xi &= c(t,x),\\
\eta^1 &= q^1(t)v + r^1(t,x)u + p^1(t,x),\\
\eta^2 &= q^2(t)u + r^2(t,x)v + p^2(t,x),
\end{aligned} \tag{18}
$$

where the functions on the right-hand side are arbitrarily those of their arguments.

The remaining equations (8)–(13) of the system (16) involve the unknown functions C^1 and C^2 and are called the classification equations. To solve them one should consider three different cases depending on the functions $q^1(t), q^2(t)$ and $\xi(t,x)$ arising in (18):

(a) $q^1(t) = q^2(t) = 0, \quad \xi(t,x) \neq 0;$
(b) $q^1(t) = q^2(t) = 0, \quad \xi(t,x) = 0;$
(c) $q^1(t)^2 + q^2(t)^2 \neq 0, \quad \xi(t,x) = 0.$

Note that the fourth possible case $q^1(t)^2 + q^2(t)^2 \neq 0, \xi(t,x) \neq 0$ arises only under the restriction $m = n = 0$, which follows from (8) to (9) of system (16). This means the case

Table 1. *Q*-conditional symmetries of the RD system (3).

No	RD systems of the form (1)	*Q*-conditional operators	Restrictions
1	$U_t = (U^k U_x)_x + f(U^{k+1})$, $V_t = (V^{-\frac{1}{2}} V_x)_x - 2\lambda V^{\frac{1}{2}} + g(U^{k+1})$	$\partial_t + 2p(x)V^{\frac{1}{2}}\partial_V$	$p_{xx} = (p)^2 + \lambda p$, $p \neq 0$
2	$U_t = (U^k U_x)_x + \lambda_1 U^{-k} + f(U^{k+1} - \alpha V^{l+1})$, $V_t = (V^l V_x)_x + \lambda_2 V^{-l} + g(U^{k+1} - \alpha V^{l+1})$	$\partial_t + \lambda_1 U^{-k}\partial_U$ $+\lambda_2 V^{-l}\partial_V$	$\alpha = \frac{\lambda_1(k+1)}{\lambda_2(l+1)}, \lambda_2 \neq 0$ $\lambda_1^2 + l^2 \neq 0$.
3	$U_t = (U^{-\frac{1}{2}} U_x)_x - 2\lambda U^{\frac{1}{2}} + f(U^{\frac{1}{2}} - V^{\frac{1}{2}})$, $V_t = (V^{-\frac{1}{2}} V_x)_x - 2\lambda V^{\frac{1}{2}} + g(U^{\frac{1}{2}} - V^{\frac{1}{2}})$	∂_t $+2p(x)(U^{\frac{1}{2}}\partial_U + V^{\frac{1}{2}}\partial_V)$	$p_{xx} = (p)^2 + \lambda p$, $p \neq 0$
4	$U_t = (U^k U_x)_x + \lambda_1 U^{-k} + f(\omega)$, $V_t = (V^l V_x)_x + (V^{l+1} - \lambda_3)(g(\omega) + \lambda_2 V^{-l})$	$\partial_t + \lambda_1 U^{-k}\partial_U$ $+\lambda_2(V - \lambda_3 V^{-l})\partial_V$	$\omega = \frac{\exp U^{k+1}}{(V^{l+1}-\lambda_3)^{\frac{\lambda_1(k+1)}{\lambda_2(l+1)}}}, \lambda_2 \neq 0$, either $\lambda_1^2 + \lambda_3^2 \neq 0$ or $\lambda_3^2 + k^2 \neq 0$ or $\lambda_1^2 + l^2 \neq 0$
5	$U_t = (U^k U_x)_x + (U^{k+1} - \lambda_1)(f(\omega) + \lambda_2 U^{-k})$ $V_t = (V^l V_x)_x + (V^{l+1} - \lambda_3)(g(\omega) + \lambda_4 V^{-l})$	$\partial_t + \lambda_2(U - \lambda_1 U^{-k})\partial_U$ $+\lambda_4(V - \lambda_3 V^{-l})\partial_V$	$\omega = \frac{U^{k+1}-\lambda_1}{(V^{l+1}-\lambda_3)^{\frac{\lambda_2(k+1)}{\lambda_4(l+1)}}}, \lambda_2\lambda_4 \neq 0$, either $\lambda_1^2 + \lambda_3^2 \neq 0$, or $\lambda_3^2 + k^2 \neq 0$ or $\lambda_1^2 + l^2 \neq 0$

of the RD system (3) with constant diffusivities should be separately considered so that we assume $m^2 + n^2 \neq 0$ below.

Solving equations (8)–(13) of system (16) in case (a) is rather simple but very cumbersome. After the relevant computations (the program package MATHEMATICA 5.0 was also used) it has been established that all the operators obtained are nothing but the Lie symmetry operators (up to the relevant multiplier $M(t, x, U, V)$) found earlier in [8]. In contrary to (a), the case (c) is the most difficult one and at present, we were able to solve equations (8)–(13) of system (16) only in particular subcases.

Now we present the theorem giving a complete description of *Q*-conditional symmetry operators of the form (4) in case (b).

Theorem 1. *The RD system (3) with $k^2 + l^2 \neq 0$ and $(k + 1)(l + 1) \neq 0$ is Q-conditional invariant under the operator (4) with $\xi = 0$ and $\eta^1_V = \eta^2_U = 0$ if and only if it and the relevant operator (up to equivalent representations generated by multiplying on the arbitrary smooth function $M(t, x, U, V)$) have the forms listed in table 1 (here f and g are arbitrary smooth functions of the relevant argument, while λ_j, $j = 1, 2, 3, 4$, are arbitrary constants).*

Sketch of the proof of the theorem. To prove the theorem one needs to construct the general solution of subsystem (8)–(13) of system (16) under the assumption that $\xi = 0$ and $\eta^1_V = \eta^2_U = 0$. Obviously, equations (8) and (9) are automatically satisfied, while (10) and (11) are reduced to the form $\eta^1_{xU} = 0$ and $\eta^2_{xU} = 0$, respectively, i.e.

$$r^1 = r^1(t), \qquad r^2 = r^2(t).$$

So, equations (12) and (13) take the form

$$\begin{aligned}
&(r^1u + p^1)C^1_u + (r^2v + p^2)C^1_v - r^1C^1 \\
&\qquad + \left(r^1_t + m(r^1)^2\right)u^{m+1} + \left(p^1_t + 2mr^1p^1\right)u^m + m(p^1)^2u^{m-1} - p^1_{xx} = 0, \\
&(r^1u + p^1)C^2_u + (r^2v + p^2)C^2_v - r^2C^2 \\
&\qquad + \left(r^2_t + n(r^2)^2\right)v^{n+1} + \left(p^2_t + 2nr^2p^2\right)v^n + n(p^2)^2v^{n-1} - p^2_{xx} = 0.
\end{aligned} \tag{19}$$

System (19) consists of two independent first-order linear PDEs with respect to the unknown functions $C^1(u, v)$ and $C^2(u, v)$; therefore its general solution can be straightforwardly constructed, however we should remember that the coefficients in (19) are functions on t and x. To construct all the possible solutions of (19) one needs to consider six different cases (up to renaming $u \to v$ and $v \to u$):

(1) $r^1 = p^1 = r^2 = p^2 = 0$,
(2) $r^1 = p^1 = 0$, $r^2 \neq 0$,
(3) $r^1 = p^1 = r^2 = 0$, $p^2 \neq 0$,
(4) $r^1 = 0$, $p^1 \neq 0$, $r^2 = 0$, $p^2 \neq 0$,
(5) $r^1 = 0$, $p^1 \neq 0$, $r^2 \neq 0$,
(6) $r^1 \neq 0$, $r^2 \neq 0$.

In case (1) operator (4) immediately takes the form $Q = \partial_t$, which is, of course, the Lie symmetry operator. The similar situation occurs in case (2) since all the operators obtained are equivalent to the relevant Lie symmetry operators listed in [8]. The most interesting cases are (3)–(6).

Consider case (3) in detail. In this case system (19) takes the form

$$\begin{aligned} &p^2 C^1_v = 0, \\ &p^2 C^2_v + p^2_t v^n + n(p^2)^2 v^{n-1} - p^2_{xx} = 0 \end{aligned} \tag{20}$$

and its formal integration leads to the solution

$$\begin{aligned} &C^1 = f(u) \\ &C^2 = \int \left(\frac{p^2_{xx}}{p^2} - \frac{p^2_t}{p^2} v^n - np^2 v^{n-1} \right) \mathrm{d}v + g(u), \end{aligned} \tag{21}$$

where f and g are arbitrary smooth functions. Since the function C^2 does not depend on t and x, three subcases should be separately examined: $n = 0, n = 1$ and $n \neq 0; 1$. The first subcase immediately gives $C^2 = \frac{p^2_{xx} - p^2_t}{p^2} v + g(u)$, so that

$$\frac{p^2_{xx} - p^2_t}{p^2} = \lambda,$$

where λ is an arbitrary constant. So, the system

$$u_{xx} = u^m u_t + f(u), \qquad v_{xx} = v_t + \lambda v + g(u) \tag{22}$$

admits the Q-conditional symmetry operator

$$Q = \partial_t + p^2(t, x)\partial_v, \tag{23}$$

where $p^2(t, x)$ is the general solution of the linear PDE $p^2_t = p^2_{xx} - \lambda p^2$. However, if one now applies substitution (9) to (22) and (23), then the RD system and the operator listed in [8] (see case (5) in table 1) are obtained. So, subcase $n = 0$ does not lead to any Q-conditional symmetries.

In the subcase $n = 1$ the general solution of (20) takes the form

$$C^1 = f(u), \qquad C^2 = \lambda v + g(u),$$

where $\lambda = \frac{p^2_{xx}}{p^2} - p^2$. So, the system

$$u_{xx} = u^m u_t + f(u), \qquad v_{xx} = vv_t + \lambda v + g(u) \tag{24}$$

admits the Q-conditional symmetry operator

$$Q = \partial_t + p^2(x)\partial_v, \tag{25}$$

where the function $p^2(x)$ is the general solution of the nonlinear ODE

$$p^2_{xx} = (p^2)^2 + \lambda p^2. \tag{26}$$

Applying now substitution (9) to (24)–(25) and introducing the relevant notation, one arrives at the system and the Q-conditional operator listed in case (1) of table 1.

Considering the subcase $n \neq 0; 1$, we immediately obtain $p^2 = \lambda = \text{const}$ (see (21)) and this leads to the system

$$u_{xx} = u^m u_t + f(u), \qquad v_{xx} = v^n v_t - \lambda v^n + g(u) \tag{27}$$

and the operator

$$Q = \partial_t + \lambda \partial_v. \tag{28}$$

Operator (28) is reduced to the form $Q = \partial_t + \frac{1}{l+1}\lambda V^{-l}\partial_V$ with $\lambda \neq 0$ by using substitution (9). On the other hand, system (27) and operator (28) correspond to a particular case at $\lambda_1 = \alpha = 0$ of those listed in case (2) of table 1. Thus, case (3) is completely investigated.

Case (4) can be examined in a quite similar way and the system

$$\begin{aligned} u_{xx} &= u^m u_t - \alpha\lambda u^m + f(u - \alpha v), \\ v_{xx} &= v^n v_t - \lambda v^n + g(u - \alpha v), \\ Q &= \partial_t + \lambda(\alpha\partial_u + \partial_v), \end{aligned} \tag{29}$$

and the operator

$$Q = \partial_t + \lambda(\alpha\partial_u + \partial_v) \tag{30}$$

is obtained, where $\alpha \neq 0$ is an arbitrary constant. It is easily seen that systems and operators (27)–(30) can be united, i.e., the restriction $\alpha \neq 0$ is not essential. Applying now substitution (9) to (29)–(30) and introducing the relevant notations, one arrives at the system and the Q-conditional operator listed in case (2) of table 1. It turns out that the power $m = n = 1$ leads to an additional symmetry in this case. In fact, the system

$$u_{xx} = uu_t + \lambda u + f(u - v), \qquad v_{xx} = vv_t + \lambda v + g(u - v) \tag{31}$$

is conditionally invariant with respect to the operator

$$Q = \partial_t + p^2(x)(\partial_u + \partial_v), \tag{32}$$

where $p^2(x)$ is the general solution of the nonlinear ODE (26). Formulae (31)–(32) together with the substitution (9) generate the RD system and the operator listed in case (3) of table 1. Those listed in cases 4 and 5 of the table can be similarly obtained by the examination of the cases (5) $r^1 = 0$, $p^1 \neq 0$, $r^2 \neq 0$ and (6) $r^1 \neq 0$, $r^2 \neq 0$.

Finally, we note that all operators arising in table 1 are not Lie symmetry operators because any Lie symmetry operator of the RD system (3) must be linear on U and V [8].

The sketch of the proof is now completed.

Remark 1. Restrictions on the coefficients λ_k arising in the last column of table 1 guarantee that the relevant operators do not coincide with Lie symmetries. Of course, those operators are still Q-conditional symmetry operators if some λ's vanish and then they are equivalent to the relevant Lie symmetry operators obtained in [8].

3. Exact solutions and their application

In this section, we apply the operators of conditional symmetry for constructing exact solutions of the relevant RD systems, demonstrate their application for solving boundary value problems and present some biological interpretation.

Table 2. Ansätze and reduced systems of ODEs for RD systems of the form (3).

No	Ansätze	Systems of ODEs
1	$U = (\varphi(x))^{\frac{1}{k+1}}$ $V = (p^2 t + \psi(x))^2$	$\varphi_{xx} = -(k+1)f(\varphi)$ $\psi_{xx} = -\frac{1}{2}g(\varphi) + (\lambda + p)\psi$
2	$U = (\lambda_1(k+1)t + \varphi(x))^{\frac{1}{k+1}}$ $V = (\lambda_2(l+1)t + \psi(x))^{\frac{1}{l+1}}$	$\varphi_{xx} = -(k+1)f(\varphi - \alpha\psi)$ $\psi_{xx} = -(l+1)g(\varphi - \alpha\psi)$
3	$U = (p^2 t + \varphi(x))^2$ $V = (p^2 t + \psi(x))^2$	$\varphi_{xx} = (p+\lambda)\varphi - \frac{1}{2}f(\varphi - \psi)$ $\psi_{xx} = (p+\lambda)\psi - \frac{1}{2}g(\varphi - \psi)$
4	$U = (\lambda_1(k+1)t + \varphi(x))^{\frac{1}{k+1}}$ $V = (\exp(\lambda_2(l+1)t + \psi(x)) + \lambda_3)^{\frac{1}{l+1}}$	$\varphi_{xx} = -(k+1)f\left(\exp(\varphi - \frac{\lambda_1(k+1)}{\lambda_2(l+1)}\psi)\right)$ $\psi_{xx} + \psi_x^2 = -(l+1)g\left(\exp(\varphi - \frac{\lambda_1(k+1)}{\lambda_2(l+1)}\psi)\right)$
5	$U = (\varphi(x)\exp(\lambda_2(k+1)t) + \lambda_1)^{\frac{1}{k+1}}$ $V = (\psi(x)\exp(\lambda_4(l+1)t) + \lambda_3)^{\frac{1}{l+1}}$	$\varphi_{xx} = -(k+1)\varphi f\left(\varphi\psi^{-\frac{\lambda_2(k+1)}{\lambda_4(l+1)}}\right)$ $\psi_{xx} = -(l+1)\psi g\left(\varphi\psi^{-\frac{\lambda_2(k+1)}{\lambda_4(l+1)}}\right)$

It is well known that any Q-conditional symmetry operator of the given two-dimensional PDE (system of PDEs) guarantees its reduction to an ODE (system of ODEs) by the construction of the corresponding local substitution, which is called ansatz. Usually the ODE (system of ODEs) obtained can be integrated (at least partly) and exact solutions of the initial PDE (system of PDEs) are constructed. Nevertheless, this procedure contains new difficulties, one can successfully realize one for the operators arising in table 1. We present the final result in the form of table 2.

Remark 2. The numbers $1, \ldots, 5$ in table 2 correspond to the RD systems arising in table 1 with the same numbers.

Now one sees that the reduced systems of ODEs are nonlinear and it is quite implausible that those are integrable for arbitrary smooth functions f and g. However, these systems can be integrated if the functions f and g are correctly specified. For example, the ODE system arising in case (5) takes the form

$$\varphi_{xx} = \alpha_1\varphi + \beta_1\psi, \qquad \psi_{xx} = \alpha_2\varphi + \beta_2\psi, \tag{33}$$

if one sets $g = -\frac{1}{l+1}\left(\alpha_2\frac{\varphi}{\psi} + \beta_2\right)$, $f = -\frac{1}{k+1}\left(\alpha_1 + \beta_1\frac{\psi}{\varphi}\right)$, $r = -\lambda_2(k+1) = -\lambda_4(l+1)$, where $\alpha_j, \beta_j, j = 1, 2$, are arbitrary constants. Nevertheless the initial RD system

$$\begin{aligned} U_t &= (U^k U_x)_x - \frac{1}{k+1}(rU + \alpha_1(U^{k+1} - \lambda_1) + \beta_1(V^{l+1} - \lambda_3) - \lambda_1 r U^{-k}), \\ V_t &= (V^l V_x)_x - \frac{1}{l+1}(rV + \alpha_2(U^{k+1} - \lambda_1) + \beta_2(V^{l+1} - \lambda_3) - \lambda_3 r V^{-l}) \end{aligned} \tag{34}$$

is still nonlinear, the reducing system (33) is linear and its general solution can be constructed in an explicit form. Assuming $\beta_1 \neq 0$ (otherwise should be $\alpha_2 \neq 0$ and one will start from the second equation of (33)) the function ψ can be expressed from the first equation of (33) so that the second equation takes the form

$$\varphi_{xxxx} - (\alpha_1 + \beta_2)\varphi_{xx} + (\alpha_1\beta_2 - \alpha_2\beta_1)\varphi = 0. \tag{35}$$

The general solution of the fourth-order ODE (35) essentially depends on the roots of the algebraic equation

$$s^4 - (\alpha_1 + \beta_2)s^2 + (\alpha_1\beta_2 - \alpha_2\beta_1) = 0. \tag{36}$$

Generally speaking, nine different forms of the function φ can be obtained. To avoid cumbersome computations, we consider only the case when (36) possesses four different complex roots with the zero real parts, i.e.:

$$\begin{aligned} s_{1,2} &= \pm \mathrm{i}\sqrt{\frac{-(\alpha_1 + \beta_2 + \sqrt{(\alpha_1 - \beta_2)^2 + 4\alpha_2\beta_1})}{2}}, \\ s_{3,4} &= \pm \mathrm{i}\sqrt{\frac{-(\alpha_1 + \beta_2 - \sqrt{(\alpha_1 - \beta_2)^2 + 4\alpha_2\beta_1})}{2}}, \qquad \mathrm{i}^2 = -1. \end{aligned} \tag{37}$$

It occurs under the following restrictions on the coefficient of (35):

$$(\alpha_1 - \beta_2)^2 > -4\alpha_2\beta_1, \qquad \alpha_1 + \beta_2 < -\sqrt{(\alpha_1 - \beta_2)^2 + 4\alpha_2\beta_1}. \tag{38}$$

Hence, we obtain the general solution of (33):

$$\begin{aligned} \varphi &= A_1\cos(|s_1|x) + A_2\sin(|s_1|x) + A_3\cos(|s_3|x) + A_4\sin(|s_3|x), \\ \psi &= -\frac{1}{\beta_1}((\alpha_1 + |s_1|^2)(A_1\cos(|s_1|x) + A_2\sin(|s_1|x)) \\ &\quad + (\alpha_1 + |s_3|^2)(A_3\cos(|s_3|x) + A_4\sin(|s_3|x))), \end{aligned} \tag{39}$$

where A_j, $j = 1, 2, 3, 4$, are arbitrary constants. Substituting (39) into the ansatz

$$\begin{aligned} U &= (\varphi(x)\exp(\lambda_2(k+1)t) + \lambda_1)^{\frac{1}{k+1}}, \\ V &= (\psi(x)\exp(\lambda_4(l+1)t) + \lambda_3)^{\frac{1}{l+1}}, \end{aligned} \tag{40}$$

arising in case (5) of table 2, we arrive at the fourth-parametrical family of exact solutions of the nonlinear RD system (34) with the coefficients satisfying restrictions (38). It should be noted that all solutions of the form (39)–(40) are periodic on the space variable x. According to Murray [2], the periodic (in space) solutions can mathematically express the Turing instability, which leads to the spatially inhomogeneous structures and forms. Moreover, we can show that solutions of the form (39)–(40) with the correctly specified constants A_j, $j = 1, \dots, 4$, satisfy the zero-Neumann conditions (zero flux boundary conditions), which naturally arise in mathematical models with the Turing instability. In fact, consider the space interval $[0, a], a > 0$ where the cells with the density U and chemicals with the density V are interacting. Assuming that their interaction is described by the RD system (34), the zero-flux boundary conditions on $[0, a]$ lead to the requirements

$$\begin{aligned} U_x|_{x=0} &= 0, \qquad V_x|_{x=0} = 0, \\ U_x|_{x=a} &= 0, \qquad V_x|_{x=a} = 0. \end{aligned} \tag{41}$$

Using ansatz (40), one obtains formulae

$$\begin{aligned} U_x &= \frac{1}{k+1}\varphi_x(\varphi\exp(-rt) + \lambda_1)^{-\frac{k}{k+1}}, \\ V_x &= \frac{1}{l+1}\psi_x(\psi\exp(-rt) + \lambda_3)^{-\frac{l}{l+1}}. \end{aligned} \tag{42}$$

Hence, taking into account solution (39) and conditions (41) with $x = 0$, we arrive at two linear algebraic equations

$$\begin{aligned} \varphi_x|_{x=0} &= A_2|s_1| + A_4|s_3| = 0, \\ \psi_x|_{x=0} &= A_2|s_1|(\alpha_1 + |s_1|^2) + A_4|s_3|(\alpha_1 + |s_3|^2) = 0 \end{aligned}$$

to find A_2 and A_4. Analyzing these equations one easily finds that the solution $A_2 = A_4 = 0$ is only possible. Finally, using conditions (41) with $x = a$, we obtain

$$\begin{aligned} &A_1|s_1|\sin(|s_1|a) + A_3|s_3|\sin(|s_3|a) = 0, \\ &A_1|s_1|(\alpha_1 + |s_1|^2)\sin(|s_1|a) + A_3|s_3|(\alpha_1 + |s_3|^2)\sin(|s_3|a) = 0 \end{aligned} \tag{43}$$

to find A_1 and A_3. Analyzing (43), we deduce that the space interval $[0, a]$, $a > 0$, cannot be arbitrary otherwise $\varphi = \psi = 0$. There are three countable sets of values for a: (i) $a = \frac{\pi j_1}{|s_1|}$, (ii) $a = \frac{\pi j_1}{|s_3|}$ and (iii) $a = \frac{\pi\sqrt{j_1^2+j_2^2}}{\sqrt{-(\alpha_1+\beta_2)}}$, $(j_1, j_2) \subset \mathbb{N}^2$, for which condition (43) is satisfied by the functions

$$\varphi = A_1\cos(|s_1|x), \qquad \psi = -\frac{1}{\beta_1}A_1(\alpha_1 + |s_1|^2)\cos(|s_1|x),$$

$$\varphi = A_3\cos(|s_3|x), \qquad \psi = -\frac{1}{\beta_1}A_3(\alpha_1 + |s_3|^2)\cos(|s_3|x),$$

and

$$\varphi = A_1\cos(|s_1|x) + A_3\cos\left(\frac{j_1}{j_2}|s_1|x\right),$$

$$\psi = -\frac{1}{\beta_1}\left(A_1(\alpha_1 + |s_1|^2)\cos(|s_1|x) + A_3\left(\alpha_1 + \frac{j_1^2}{j_2^2}|s_1|^2\right)\cos\left(\frac{j_1}{j_2}|s_1|x\right)\right),$$

respectively. Moreover, the additional restriction $\alpha_2 = \frac{(\alpha_1 j_2^2 - \beta_2 j_1^2)(\beta_2 j_2^2 - \alpha_1 j_1^2)}{(j_1^2+j_2^2)^2\beta_1}$ is obtained in the case (iii), which allows us to simplify the expression for $|s_1|$ and $|s_3|$:

$$|s_1| = \frac{j_2\sqrt{-(\alpha_1 + \beta_2)}}{\sqrt{j_1^2 + j_2^2}}, \qquad |s_3| = \frac{j_1}{j_2}|s_1|. \tag{44}$$

Thus, we can now formulate the theorem.

Theorem 2. *The nonlinear RD system (34) with the coefficients satisfying restrictions (38) possesses the periodic (in space) solutions*

$$\begin{aligned} &(i)\ U = (A_1\cos(|s_1|x)\exp(-rt) + \lambda_1)^{\frac{1}{k+1}}, \\ &V = \left(-(A_1(\alpha_1 + |s_1|^2)\cos(|s_1|x))\frac{\exp(-rt)}{\beta_1} + \lambda_3\right)^{\frac{1}{l+1}}, \end{aligned} \tag{45}$$

satisfying the zero-boundary conditions (41) on the interval $\left[0, \frac{\pi j_1}{|s_1|}\right]$, $j_1 \in \mathbb{N}$;

$$\begin{aligned} &(ii)\ U = (A_3\cos(|s_3|x)\exp(-rt) + \lambda_1)^{\frac{1}{k+1}}, \\ &V = \left(-(A_3(\alpha_1 + |s_3|^2)\cos(|s_3|x))\frac{\exp(-rt)}{\beta_1} + \lambda_3\right)^{\frac{1}{l+1}}, \end{aligned} \tag{46}$$

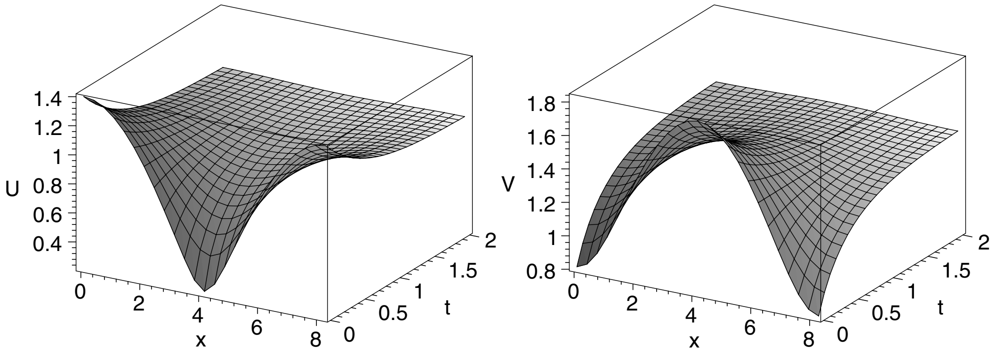

Figure 1. Exact solution (45) with $r = 2$, $A_1 = 0.95$, $s_1 = \sqrt{2-\sqrt{2}}$, $\lambda_1 = 1$, $\lambda_3 = 2$.

satisfying these conditions on the interval $\left[0, \frac{\pi j_1}{|s_3|}\right]$, $j_1 \in \mathbb{N}$;

(iii) $U = \left(\left(A_1\cos(|s_1|x) + A_3\cos\left(\frac{j_1}{j_2}|s_1|x\right)\right)\exp(-rt) + \lambda_1\right)^{\frac{1}{k+1}}$,

$$V = \left(-\left(A_1(\alpha_1 + |s_1|^2)\cos(|s_1|x) + A_3\left(\alpha_1 + \frac{j_1^2}{j_2^2}|s_1|^2\right)\cos\left(\frac{j_1}{j_2}|s_1|x\right)\right)\frac{\exp(-rt)}{\beta_1} + \lambda_3\right)^{\frac{1}{l+1}}, \tag{47}$$

satisfying these conditions on the interval $\left[0, \frac{\pi\sqrt{j_1^2+j_2^2}}{\sqrt{-(\alpha_1+\beta_2)}}\right]$, $(j_1, j_2) \subset \mathbb{N}^2$.

In cases (i) and (ii), the values of $|s_1|$ *and* $|s_3|$ *are determined by formulae (37), in the case (iii) the value of* $|s_1|$ *is given in (44) and* $\alpha_2 = \frac{(\alpha_1 j_2^2 - \beta_2 j_1^2)(\beta_2 j_2^2 - \alpha_1 j_1^2)}{(j_1^2+j_2^2)^2\beta_1}$.

Remark 3. Since the system (34) and the boundary conditions (41) are invariant under the space translation transformations $x \to x - x_0$, where x_0 is an arbitrary parameter, one can generalize this theorem on the intervals $[-a, 0]$ and $[-a, a]$.

One sees that each solution of the forms (45)–(47) tends to the steady-state point $(U, V) = ((\lambda_1)^{\frac{1}{k+1}}, (\lambda_3)^{\frac{1}{l+1}})$ if the time $t \to +\infty$ and $r > 0$. This point can be stable or unstable depending on the coefficients of the system (34). An example of solution (45) of the RD system

$$\begin{aligned} U_t &= (UU_x)_x - \frac{r}{2}U + U^2 + \frac{1}{2}V^2 + \frac{r-4U}{2U}, \\ V_t &= (VV_x)_x - \frac{r}{2}V + U^2 + V^2 + \frac{r-3V}{V} \end{aligned} \tag{48}$$

with $r = 2$ is presented in figure 1. This solution arrives at the steady-state point $(U, V) = (1, \sqrt{2})$ very fast.

Solutions (45)–(47) with $r < 0$ are growing to infinity with time providing $k+1 > 0$ and $l+1 > 0$. On the other hand, the spatially inhomogeneous structures should be formed for a finite time (the development of those structures in an organism should be finished during the organism life). Taking this into account, one notes that solutions (45)–(47) with $r < 0$ may describe the development of spatially inhomogeneous structures for the finite time $t_{\max}$. The relevant example of solution (45) of the RD system (48) with $r = -2$ is presented in figure 2.

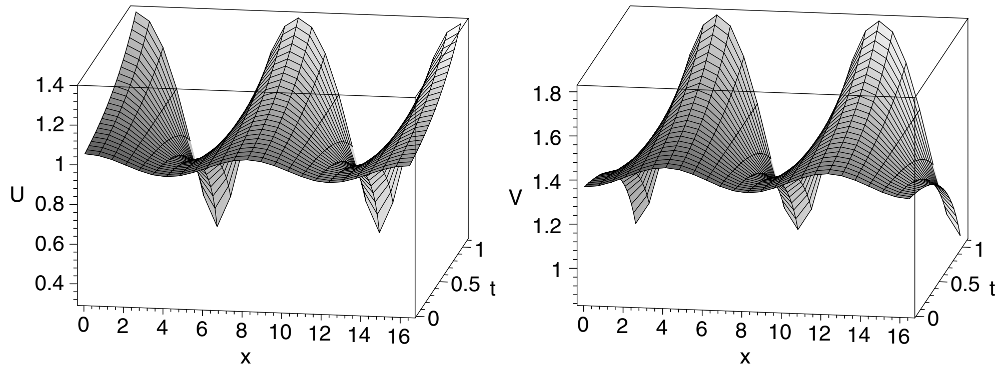

Figure 2. Exact solution (45) with $r = -2$, $A_1 = 0.1$, $s_1 = \sqrt{2 - \sqrt{2}}$, $\lambda_1 = 1$, $\lambda_3 = 2$.

One sees that both components, U and V, have practically homogeneous distributions in space if $t \ll t_{\max}$. However, those are essentially inhomogeneous if $t \gg 0$. Moreover each of the components U and V dominates in the different regions. So, we may say that this solution forms a striped structure. This kind of spatially inhomogeneous structure is typical for mammalian coats (tigers, zebras).

4. Conclusions

In this paper, theorem 1 giving a complete description of Q-conditional symmetries of the nonlinear RD system (3) under some restrictions is proved. We established that there are a wide range of nonlinear RD systems of five types, which are conditionally invariant with respect to the operator (4) with the coefficients $\xi = 0$ and $\eta^1_V = \eta^2_U = 0$. Moreover those systems do not coincide with the RD systems admitting a nontrivial Lie symmetry, which has been completely described in [7, 8]. To the best of our knowledge, there are no such papers devoted to constructions of conditional symmetries for *systems of evolution equations*, excepting the recent paper [26]. Note that only an example of the nonlinear diffusion–convection system was found in [26], while here we present five classes of the nonlinear RD systems and each of them involves two arbitrary functions. The work is in progress to solve the problem without any restrictions on the coefficients of operator (4).

The Q-conditional symmetry operators have been applied for obtaining exact solutions of the relevant RD systems. In fact, we reduced the problem of solving PDE systems to the one for the relevant ODE systems. These ODE systems were further solved, therefore several exact solutions in explicit form have been constructed for the RD systems (34). Properties of the solutions obtained were investigated and a possible biological interpretation was suggested.

Acknowledgments

The authors are grateful to the unknown board member and the unknown referee for the useful comments.

References

[1] Turing A M 1952 The chemical basis of morphogenesis *Phil. Trans. R. Soc.* A **237** 37–72
[2] Murray J D 1989 *Mathematical Biology* (Berlin: Springer)

[3] Britton N F 2003 *Essential Mathematical Biology* (Berlin: Springer)

[4] Cherniha R and King J R 2000 Lie symmetries of nonlinear multi-dimensional reaction–diffusion systems: I *J. Phys. A: Math. Gen.* **33** 267–82, 7839–41

[5] Cherniha R and King J R 2003 Lie symmetries of nonlinear multi-dimensional reaction–diffusion systems: II *J. Phys. A: Math. Gen.* **36** 405–25

[6] Nikitin A G 2005 Group classification of systems of nonlinear reaction–diffusion equations *Ukr. Math. Bull.* **2** 153–204

[7] Knyazeva I V and Popov M D 1994 A system of two diffusion equations *Lie Group Analysis of Differential Equations* vol 1, ed N H Ibragimov (Boca Raton, FL: Chemical Rubber Company (CRC Press)) pp 171–6

[8] Cherniha R and King J R 2005 Nonlinear reaction–diffusion systems with variable diffusivities: Lie symmetries, ansätze and exact solutions *J. Math. Anal. Appl.* **308** 11–35

[9] Cherniha R and King J R 2006 Lie symmetries and conservation laws of nonlinear multi-dimensional reaction–diffusion systems with variable diffusivities *IMA J. Appl. Math.* **71** 391–408

[10] Fushchych W I, Shtelen W M and Serov M I 1993 *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics* (Dordrecht: Kluwer)

[11] Serov M I 1990 Conditional invariance and exact solutions of nonlinear-heat equation *Ukr. Math. J.* **42** 1370–6

[12] Nucci M C 1992 Symmetries of linear, *C*-integrable, *S*-integrable and nonintegrable equations and dynamical systems *Nonlinear Evolution Equations and Dynamical Systems* (River Edge, NJ: World Scientific) pp 374–81

[13] Clarkson P A and Mansfield E L 1993 Symmetry reductions and exact solutions of a class of nonlinear heat equations *Physica* D **70** 250–88

[14] Arrigo D J, Broadbridge P and Hill J M 1994 Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source *IMA J. Appl. Math.* **52** 1–24

[15] Arrigo D J and Hill J M 1995 Nonclassical symmetries for nonlinear diffusion and absorption *Stud. Appl. Math.* **94** 21–39

[16] Pucci E and Saccomandi G 2000 Evolution equations, invariant surface conditions and functional separation of variables *Physica* D **139** 28–47

[17] Cherniha R and Serov M 1998 Symmetries, ansätze and exact solutions of nonlinear second-order evolution equations with convection term *Eur. J. Appl. Math.* **9** 527–42

[18] Cherniha R 2007 New *Q*-conditional symmetries and exact solutions of some reaction–diffusion–convection equations arising in mathematical biology *J. Math. Anal. Appl.* **326** 783–99

[19] Cherniha R and Pliukhin O 2007 New conditional symmetries and exact solutions of nonlinear reaction–diffusion–convection equations *J. Phys. A: Math. Gen.* **40** 10049–70

[20] Dixon J M, Tuszynski J A and Clarkson P A 1997 *From Nonlinearity to Coherence* (Oxford: Clarendon)

[21] Bradshaw-Hajek B H and Broadbridge P 2004 A robust cubic reaction–diffusion system for gene propagation *Math. Comput. Modelling* **39** 1151–63

[22] Okubo A and Levin S A 2001 *Diffusion and Ecological Problems. Modern Perspectives* 2nd ed (Berlin: Springer)

[23] Bluman G W and Cole J D 1969 The general similarity solution of the heat equation *J. Math. Mech.* **18** 1025–42

[24] Fushchych W I, Serov M I and Chopyk W 1988 Conditional invariance and nonlinear heat equations *Dopov. Akad. Nauk Ukr. RSR (Proc. Ukr. Acad. Sci. Ser. A)* A **9** 17–21 (in Ukrainian)

[25] Olver P 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

[26] Cherniha R and Serov M 2003 Nonlinear systems of the burgers-type equations: Lie and *Q*- conditional symmetries, ansätze and solutions *J. Math. Anal. Appl.* **282** 305–28